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INEQUALITIES

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Summary: A speech prepared for delivery before a Teachers' Conference at UCLA, 8 July 1953, containing an elementary presentation of some fundamental inequalities.

INEQUALITIES

Richard Bellman

It has been said that mathematics is the science of tautology, which is to say that mathematicians spend their time proving that equal quantities are equal. This statement is wrong on two counts: In the first place, mathematics is not a science, it is an art; in the second place, it is fundamentally the study of inequalities rather than equalities.

I would like today to discuss a number of the basic inequalities of analysis, presenting first an algebraic proof of the inequality between the arithmetic and geometric means, and then a most elegant geometric technique due to Young. In passing we will observe how the theory of inequalities may be used to supplant the calculus in many common types of maximization and minimization problems. Finally, I shall show how Young's inequality leads naturally to Holder's inequality, and Holder's inequality to Minkowski's.

Since it has become unfashionable in educational circles to pose problems in the spirit of a spelling bee, but rather to motivate the student by relating the problem to our everyday pursuits, we shall consider the following question which is perhaps typical of the way in which the theory of inequalities can enter into our ordinary pursuits.

A football player of some renown, having gone into stocks and bonds and made a substantial score there also, stipulated in his will that his coffin be enclosed in a giant football. His executors, of an economical turn of mind, were confronted with the problem of finding the dimensions of the smallest football which would meet the terms of the will.

Indulging in the usual mathematical license, we may consider the football to be an ellipsoid and reduce the problem to that of finding the coffin of maximum volume which will fit into a given ellipsoid.

Taking the ellipsoid to have the equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, we see the analytical equivalent of the practical problem above is that of finding the maximum of $v = 8xyz$ subject to the above constraint on x, y and z.

Let us first observe that we can simplify by observing that v and $v^2/a^2b^2c^2$ are maximized simultaneously. Hence replacing x^2/a^2 by u, y^2/b^2 by v, z^2/c^2 by w, the problem reduces to maximizing uvw subject to $u + v + w = 1$, $u,v,w \geq 0$. It is intuitive now that the symmetric point $u = v = w = 1/3$ should play a distinguished role, as either a minimum or a maximum. Since it is clearly not a minimum it follows that it furnishes the desired maximum.

§2. An Algebraic Approach.

In order to prove this, in a purely algebraic fashion without the aid of calculus, we shall derive a general inequality connecting the arithmetic mean of n positive quantities, $(a_1 + a_2 + \cdots + a_n)/n$, and the geometric mean, $\sqrt[n]{a_1 a_2 \cdots a_n}$, namely

$$(1)\qquad \sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality occurring only if

$$(2)\qquad a_1 = a_2 = \cdots = a_n.$$

There are literally hundreds of proofs of this basic inequality, many of which are actually quite different. The proof I will present is perhaps not the simplest, but it is one of the most ingenious. It is perhaps the only application of a particular form of mathematical induction and I think that it will be interesting for that reason.

The proof begins in a very simple manner. The most basic inequality, which I must confess is a tautology, is that a non-negative number is greater than or equal to zero. The simplest non-negative number, and one which is invariably non-negative, is a square of a number. Taking, for our own purposes (and this is where ingenuity enters), the number $a - b$ and squaring it, we have the inequality

$$(3)\qquad (a - b)^2 \ge 0$$

Multiplying out and transposing, we have the well-known inequality,

$$(4)\qquad \frac{a^2 + b^2}{2} \ge ab$$

together with the important addition that equality can occur if and only if $a = b$. Setting $a^2 = a_1$, $b^2 = b_1$, we have the well-known

result that the arithmetic mean of 2 positive quantities is greater than or equal to their geometric mean.

Let us now replace a_1 by $a_1+a_2/2$ and b_1 by $a_3+a_4/2$ obtaining

$$(5) \qquad \frac{a_1+a_2+a_3+a_4}{4} \geq \sqrt{\left(\frac{a_1+a_2}{2}\right)\left(\frac{a_3+a_4}{2}\right)}$$

and apply the separate inequalities

$$(6) \qquad \frac{a_1+a_2}{2} \geq \sqrt{a_1a_2}\,, \qquad \frac{a_3+a_4}{2} \geq a_3a_4$$

obtaining

$$(7) \qquad \frac{a_1+a_2+a_3+a_4}{4} \geq \sqrt[4]{a_1a_2a_3a_4}$$

Retracing our steps we see that we still have the important fact that equality can occur only if $a_1 = a_2 = a_3 = a_4$.

Continuing in this way we obtain, for n a power of two, the inequality

$$(8) \qquad \frac{a_1+a_2+\cdots a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n}$$

with equality occurring only if all the variables are equal.

We still cannot apply this to our problem since 3 is not a power of two. We want to show that the same inequality holds for $n = 3$. Once more we require some ingenuity. Let us take the case $n = 4$, and set

(9) $\qquad a_1 = a_1, \quad a_2 = a_2, \quad a_3 = a_3, \quad a_4 = \frac{a_1+a_2+a_3}{3}$

The resulting inequality is

(10) $\qquad \left(\frac{a_1+a_2+a_3}{3}\right) \geq \sqrt[4]{a_1 a_2 a_3 \left(\frac{a_1+a_2+a_3}{3}\right)}$

which simplifies to

(11) $\qquad \frac{a_1+a_2+a_3}{3} \geq \sqrt[3]{a_1 a_2 a_3}$

the desired inequality for three. Retracing our steps we see that equality can occur only if we have $a_1 = a_2 = a_3$.

This technique is perfectly general and yields the inequality for n – 1 whenever it has been established for n. Since we have established it for the integers 2, 4, 8, etc., we see that induction yields it for all n. Observe, however, that this is a backward induction rather than the usual forward induction.

Turning to our original problem we see that

(12) $\qquad 1/3 = \frac{u+v+w}{3} \geq \sqrt[3]{uvw}$

unless $u = v = w = 1/3$. In terms of the original variables, x, y, and z this yields

(13) $\qquad x = a/\sqrt{3}\ , \quad y = b/\sqrt{3}\ , \quad z = c/\sqrt{3}$

as the solution of our maximization problem.

Returning to (8), and taking $n = 10$, we obtain an interesting inequality by grouping the variables as follows

$$(13) \qquad \begin{aligned} &a_1 = a_2 = b_1 \\ &a_3 = a_4 = a_5 = b_2 \\ &a_6 = a_7 = a_8 = a_9 = a_{10} = b_3 \end{aligned}$$

The result is

$$(15) \qquad \frac{2b_1}{10} + \frac{3b_2}{10} + \frac{5b_3}{10} \geq b_1^{2/10}\, b_2^{3/10}\, b_3^{5/10}$$

which is a particular ca-e of the general inequality

$$(16) \qquad \left(\frac{n_1 b_1 + n_2 b_2 + \cdots n_k b_k}{n_1 + n_2 + \cdots n_k}\right)^{(n_1+n_2+\cdots+n_k)} \geq b_1^{n_1}\, b_2^{n_2} \cdots b_k^{n_k}$$

where the $n_1, n_2, \cdots, n_k$ are positive integers, $b_1, b_2, \cdots, b_k$ are positive, and equality occurs only if $b_1 = b_2 = \cdots = b_k$.

The limiting form of (16), namely

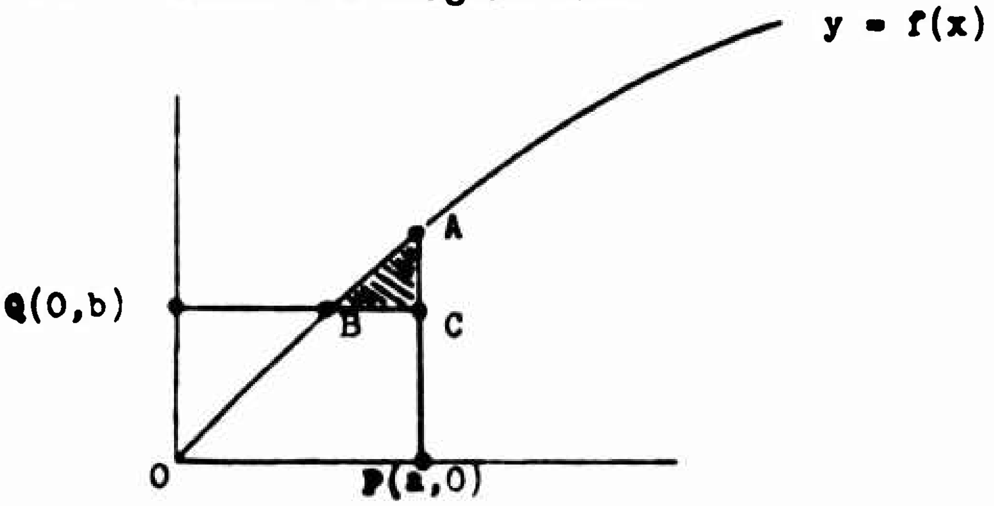
$$(17) \qquad \alpha_1 b_1 + \alpha_2 b_2 + \cdots \alpha_k b_k \geq b_1^{\alpha_1}\, b_2^{\alpha_2} \cdots b_k^{\alpha_k}$$

where $\alpha_1 \geq 0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$ is also valid, but is, of course, no longer a purely algebraic theorem.

I leave it to the reader to use the above inequality to determine the maximum of xyz subject to $x^2 + y^3 + z^5 = 1$, $x,y,z \geq 0$.

§3. A Geometric Approach.

Let us now turn to an alternate approach to the theory of inequalities. Let $f(x)$ be a monotone-increasing function with $f(0) = 0$ and consider the diagram below



The area under OAP is given by $\int_0^a f(x)dx$, while that under OQB is given by $\int_0^b f^{-1}(x)dx$ (where $f^{-1}(x)$ denotes the inverse function) clearly the sum of these two areas is greater than or equal to that of the rectangle OQCP, with equality occurring if and only if $b = f(a)$. Writing this statement in analytical terms we obtain the inequality of Young,

$$(18) \qquad \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab,$$

for $a, b \geq 0$.

Taking $f(x) = x$, we obtain (4) above. Taking $f(x) = x^{p-1}$, where $p > 1$, we obtain

$$(19) \qquad a^p/p + b^{p'}/p' \geq ab$$

where $p' = p/p-1$/(note that $1/p + 1/p' = 1$). If $f(x) = \log(1+x)$, we obtain, after some simplification

$$(20) \qquad (1+a)(\log(1+a) - (1+a) + e^{b} - b - 1 \geq ab, \qquad a,b \geq 0$$

an inequality which is important in the theory of Fourier series.

§4. Holder's Inequality.

Let us now show how (19) yields one of the most useful inequalities of analysis, the classical inequality of Holder,

$$(21) \qquad (a_1^p + a_2^p + \cdots + a_n^p)^{1/p} (b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'})^{1/p'} \geq (a_1b_1 + \cdots + a_nb_n)$$

for $a_i,\ b_i \geq 0,\quad p > 1$.

In (19) set successively

$$(22) \qquad a = a_i/(a_1^p + a_2^p + \cdots + a_n^p)^{1/p}$$

$$b = b_i/(b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'})^{1/p'}$$

and add. On the right side we obtain

$$(23) \qquad \frac{a_1b_1 + \cdots + a_nb_n}{(\qquad)^{1/p}\,(\qquad)^{1/p'}}$$

while on the left-hand side we obtain

$$(24)\quad \frac{1}{p}\left(\frac{a_1^p + a_2^p + \cdots + a_n^p}{a_1^p + \quad \cdots \quad + a_n^p}\right) + \frac{1}{p}\left(\frac{b_1^{p'} + b_2^{p'} + \cdots + b_n^{p'}}{b_1^{p'} + \cdots + b_n^{p'}}\right)$$

$$= \frac{1}{p} + \frac{1}{p'} = 1$$

This yields Holder's inequality and shows, using the condition of equality in (19) that equality holds in (21) if and only if $b_i = a_i^{p-1}$ for $i=1,2,\cdots,n$.

An alternate form of Holder's inequality which we shall use below to derive Minkowski's inequality is

$$(25)\qquad \operatorname*{Max}_{B}\,(a_1b_1 + \cdots + a_nb_n) = (a_1^p + a_2^p + \cdots + a_n^p)^{1/p}$$

where B represents the domain: $b_i \geq 0$, $b_1^{p'} + b_2^{p'} + \cdots + \cdots + \cdots + b_n^{p'} = 1$, and $q_i \geq 0$. This follows from the observation that the right-hand side is an upper bound according to Holder's inequality and is attained for $b_i = a_i^{p-1}$.

§5. Minkowski's Inequality.

Using (25) we have, for $x_k, y_k \geq 0$,

$$((x_1+y_1)^p + (x_2+y_2)^p + \cdots + (x_n+y_n)^p)^{1/p}$$

$$= \operatorname*{Max}_{B}\,((x_1+y_1)b_1 + (x_2+y_2)b_2 + \cdots + (x_n+y_n)b_n)$$

$$= \operatorname*{Max}_{B}\left[(x_1b_1 + x_2b_2 + \cdots + x_nb_n) + (y_1b_1 + y_2b_2 + \cdots + y_nb_n)\right]$$

$$\leq \operatorname*{Max}_{B}\,(x_1b_1 + x_2b_2 + \cdots + x_nb_n) + \operatorname*{Max}_{B}(y_1b_1+y_2b_2+\cdots+y_nb_n)\Big]$$

$$= (x_1^p + x_2^p + \cdots + x_n^p)^{1/p} + (y_1^p + y_2^p + \cdots + y_n^p)^{1/p}$$

We have thus established the classical inequality of Minkowski, which is for $p = p' = 2$, the famous "triangle inequality" of Euclid which states that the sum of two sides of a triangle is greater than the third. Put another way, a straight line is the shortest distance between two points.

§6. In Conclusion.

In presenting these two approaches to the theory of inequalities, I have neglected perhaps the most powerful approach, that based upon the concept of a convex function. This, however, deserves its own presentation.

For those interested in learning more about inequalities, I refer to that fascinating book by Hardy, Littlewood and Polya entitled quite simply "Inequalities."

bjc